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ORDERING THE UNIVERSE: THE ROLE OF MATHEMATICS*

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CONTENTS

1. Mathematics

Mathematics is an ancient art, and from the outset it has been both the most highly esoteric and the most intensely practical of human endeavors. As long ago as 1800 BC, the Babylonians investigated the abstract properties of numbers, and in Athenian Greece geometry attained the highest intellectual status. Alongside this theoretical understanding, mathematics blossomed as a day-to-day tool for surveying lands, for navigation, and for the engineering of public works. The practical problems and theoretical pursuits stimulated one another; it would be impossible to disentangle these two strands.

Much the same is true today. In the twentieth century, mathematics has burgeoned in scope and in diversity and has deepened in its complexity and abstraction. So profound has this explosion of research been that entire areas of mathematics may seem unintelligible to laymen—and frequently even to mathematicians working in other subfields! Despite this trend towards specialization—indeed because of it—mathematics has become more concrete and vital than ever before.

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In the past quarter century, mathematics and mathematical techniques have become an integral, pervasive, and essential component of science, technology, and business. In our technically oriented society, "innumeracy" has replaced illiteracy as our principal educational gap. One could compare the contribution of mathematics to our society with needing air and food for life. In fact, we could say that we live in the age of mathematics—that our culture has been "mathematicized." No reflection of mathematics about us is more striking than the omnipresent computer; consider a few examples of how computers influence us.

Flight. Commercial airlines can now land without a pilot's even touching the controls. Data about speed and position are relayed automatically to a device called a Kalman–Bucy filter, which flies the plane by continually finding a "least squares best fit" to a first order approximation of the laws of Newtonian physics. Similar "state filters" guide rockets and space probes and track satellites. These satellites and rockets transmit back to earth important pictures, which are "spectrally analyzed" by computer to sharpen and enhance the images.

Medicine. Large-scale sampling of data allows medical researchers to correlate disease with patterns of life style and nutrition; hence data analysis makes a general study of epidemiology possible. Computers are revolutionizing diagnosis by providing automatic blood and urine analyses as well as computer-assisted tomography (CT scans) of internal organs. Computers may soon be able to forecast medical dangers ten to twenty years in advance by running simple, noninvasive tests on a patient.

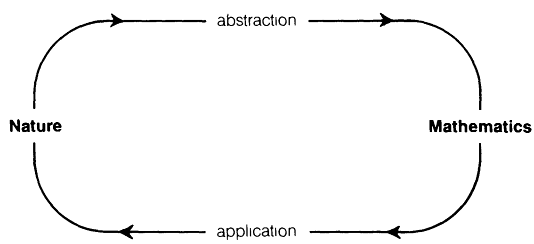
Business. The simplex method of linear programming has altered the planning of industrial production, manufacturing, inventory control, and distribution, by making it easy to compute the most efficient allocation of resources. The capacity to handle and store large blocks of data has revolutionized record keeping, billing, accounting, etc.

What do these widely different computer applications—Kalman–Bucy filters, image sharpening by spectral analyses, medical statistics, CT scanners and linear programming analyses—have in common? Each is primarily based on linear algebra, a field of mathematics that was worked out in the late 19th century with none of these applications in mind: motivation to develop this algebra came rather from an attempt to understand the geometry of n-dimensional space.

The practical implementation of these ideas occurred during this century—by people with exceptional mathematical talent. Furthermore, each of these applications involves so much data that even the fastest computers could not obtain answers by simple brute force. They also required the development and use of sophisticated mathematical techniques.

We could write several volumes to document the utilitarian value of mathematical research to our society and to show how specific mathematical ideas have influenced our world. Instead, we have chosen a few cases to illustrate the breadth and the depth of the many spinoffs from mathematics. We have a second goal, perhaps more important than simply reporting about some developments at the forefront of mathematics and science. We want to emphasize two themes that occur over and over in the history we relate.

(1) Excellent mathematics, however abstract, leads to practical applications in nature. Hard problems in nature stimulate the invention of new mathematics.



One can enter this vigorous cycle of abstraction and application from either side. The time scale from mathematics to applications varies enormously. Sometimes it is immediate; sometimes it takes a century before abstract theory causes a revolution through its application. In most cases, the time scale is somewhere in between.

(2) It is impossible to predict just where an area of mathematics will be useful. Even the originators of many mathematical ideas are often surprised by their application. The only thing we can state with certainty is that time plays tricks on anyone who claims, "There will never be any practical use for ______." The great British mathematician G. H. Hardy, for example, wrote in his autobiography, *A Mathematician's Apology,* that he practiced mathematics for its beauty, not for its practical value. He stated confidently that he saw no application whatsoever for number theory or for relativity. Only forty years later, abstract number theory has implications for national security: the properties of prime numbers are the basis for a new scheme for making secret codes. As for relativity, Hardy was disproved within just a few years—by the invention of fission and fusion devices.

It may be surprising that the most abstract subfields of mathematics—geometry, number theory, logic—have great practical importance. Computer scientist D. E. Knuth reports, "Every bit of mathematics I know has helped in some application one way or another."

Physicist Eugene Wigner marvelled at "the unreasonable effectiveness of mathematics, in the natural sciences." Surely it has something to do with the mathematician's penchant for distilling away all but a crucial aspect of a problem, for finding the common point of view from which two seemingly different problems turn out to be closely related. But this does not adequately explain why, time after time, abstract mathematics developed for its own beauty turns out, decades later, to describe nature perfectly.

Harvard mathematician Andrew Gleason has his own answer: "Mathematics is the science of order—its object is to find, describe and understand the order that underlies apparently complex situations. The principal tools of mathematics are concepts which enable us to describe this order. Precisely because mathematicians have been searching for centuries for the most efficient concepts for describing obscure instances of order, their tools are applicable to the outside world; for the real world is the very epitome of a complex situation in which there is a great deal of order."

We propose an additional reason. Mathematical ideas do not spring full-grown from the minds of researchers. History illustrates that mathematics often takes its inspiration from patterns in nature. Lessons distilled from one encounter with nature continue to serve us well when we explore other natural phenomena.

Whatever the reasons for the importance of mathematics to society, understanding how mathematics develops has crucial implications. One must assess how best to nurture excellent mathematics in this country and how to retain the world leadership gained over the past forty years. We believe in two basic principles:

Mathematical research should be as broad and as original as possible, with very long-range goals. We expect history to repeat itself; we expect that the most profound and useful future applications of mathematics cannot be predicted today, since they will arise from mathematics yet to be discovered.

While most mathematical research will be directed towards understanding known problems, we must remember that the direction of mathematics itself is constantly changing. Talented mathematicians should be encouraged to pursue research whose relevance we understand only partially or not at all, but which may ultimately result in new points of view, or in the invention of new areas of mathematics.

We have experienced a golden age of mathematics during the past forty years. Practically every subfield of mathematics has turned out, as if by magic, to be related to every other subfield, and to many applications in the natural sciences and engineering. This seamless web is not only breathtaking, it makes it impossible to be encyclopedic in describing recent mathematical research and application, confounding any overly simple organizational scheme.

Our choice of examples below is necessarily idiosyncratic, governed by our own familiarity and taste. We have organized them loosely into four areas—computation, physics, communication, and engineering—although the topics spill freely over these neat boundaries. We are aware that we are neglecting many important areas and developments. In spite of these omissions, we trust that our examples adequately reveal the nature of the mathematics as a whole.

Before we turn to these applications, we want to tell the story of a single topic in mathematics—Fourier analysis—and how it has developed in 170 years. The story illustrates how mathematics often turns out to be vastly more important than the particular problem it is invented to solve.

Fourier analysis. In the early 1800s, Jean Baptiste Joseph Fourier, newly returned from his post as civil governor of Napoleonic Egypt, set out to understand the problem of heat conduction. Given the initial temperature at all points of a region, he asked, how will heat diffuse over the course of time? It was curiosity about such phenomena as atmospheric temperature and climate that led Fourier to pose the abstract question.

In order to solve the heat diffusion equation, Fourier devised a simple—but brilliant—mathematical technique. This equation turned out to be easy to solve if the initial heat distribution were oscillatory—that is, essentially a sine wave. To take advantage of this, Fourier proposed decomposing any initial heat distribution into a (possibly infinite) sum of sine waves and then solving each of these simpler problems. The solution to the general problem could then be obtained by adding up the solutions for each of the oscillatory components, called harmonics.

French mathematicians, such as Lagrange, sharply rejected the idea, doubting that these simple harmonics could adequately express all possible functions, and casting aspersions upon Fourier's rigor. These attacks dogged Fourier for two decades, during which he carried his research forward with remarkable insight. Today we owe an enormous debt to his remarkable tenacity, his stubbornness, and his ability to proceed in spite of formidable doubts in the minds of the leaders of the scientific establishment. Fourier found it difficult to publish his work even after he received the 1811 grand prize in mathematics from the Académie des Sciences for his essay on the problem of heat conduction, because the academy's announcement of the award expressed grave reservations concerning the generality and rigor of Fourier's method. Fourier persevered and finally his work won general acceptance with the publication of his now-classic *The Analytic Theory of Heat*, in 1822.

The method of harmonic analysis, or Fourier analysis, has turned out to be tremendously important in virtually every area of mathematics and physical science—much more important than the solution of the problem of heat diffusion. In mathematics, it has become a subject by itself. But in addition the theories of differential equations, of group theory, of probability, of statistics, of geometry, of number theory, to mention a few, all use Fourier's technique for decomposing functions into their fundamental frequencies.

In physics, engineering, and computer science the effect has been no less profound. Fourier himself presaged the impact of his technique in his introduction to *The Analytic Theory of Heat:* "Profound study of nature is the most fertile course of mathematical discoveries.... It is a sure method of... discovering... the fundamental elements which are reproduced in all natural effects." In effect, Fourier provided one of the most powerful tools for mathematical physics. Once Maxwell described electromagnetic waves with his famous equations in 1873, Fourier analysis became one of the key methods for studying these waves and their harmonic components—X-rays, visible light, microwaves, radio-waves, etc. Many electrical and electronic devices are now based on Fourier analysis, including such recent ones as nuclear magnetic resonance spectrometers and X-ray crystallographic spectrometers. In this century, Fourier analysis has provided the basic understanding of quantum theory—and hence of all modern chemistry and physics.

The idea of decomposing data into periodic components has also been central in engineering. It led to the Laplace transform, taught to every engineering student as the standard method of studying linear differential equations. Fourier analysis also led to time-series analysis, which is used in oil exploration for interpreting seismic waves shot through rocks suspected of bearing petroleum.

The advent of the computer has more recently made it possible to perform Fourier analysis numerically as a routine part of data analysis. The ability to decompose sound into its harmonic components has allowed computers to generate and recognize human speech. Performing similar operations on photographs—for example, satellite pictures of regions of the Earth—allows a computer to filter out "noise" and thus sharpen or enhance the image.

Even mundane business, like the multiplication of two numbers, can be accomplished much faster by using Fourier transforms rather than the time-honored method taught in grade school. The idea is to consider the digits of the numbers as a function, which can then be expanded into a Fourier series. For 1000-digit numbers, the Fourier method may be as much as 50 times faster than the more familiar algorithm, and of course it is used in computer design.

The yeoman's duty performed by the Fourier transform is possible only because of clever methods that mathematicians discovered in order to compute the Fourier transform of a sequence of numbers—algorithms which are collectively called Fast Fourier Transforms (FFT). The idea for these is found in the work of Runge and König in 1924, although the germ of the method probably dates to Gauss' work a century earlier. The FFT became widely known and used after Cooley and Tukey's paper in 1965, and various modifications have been proposed by Garwin, Rudnick, Good, Winograd, and others.

The direct computation of the Fourier transform of n numbers requires about n^2 operations. The FFT makes it possible to find the answer in approximately $n \log n$ steps—a tremendous improvement for large values of n. Without this improvement, computers could never analyze many problems in "real time"—that is, produce answers at the same rate the data are flowing in and hence avoid bottlenecks. (Determining the exact amount of time it takes to perform the FFT turns out to be a difficult problem, which hinges on some profound theorems from analytic number theory about the distribution of prime numbers.)

At least as important as the numerous applications to science and engineering has been the application of Fourier analysis to mathematics itself. Like other scientists, mathematicians are constantly searching for new tools to solve their theoretical problems. Frequently it happens that techniques discovered to solve one abstract problem later apply to a wide variety of others.

If you need to be convinced of this, look under "Fourier" in the card catalogue of a university science library. At Harvard's, for example, there are 212 entries, of which the first ten are Fourier analysis in probability theory, Fourier analysis in several complex variables, Fourier analysis of time series, Fourier analysis of unbounded measures on locally compact abelian groups, Fourier analysis on groups and partial wave analysis, Fourier analysis of local fields, Fourier analysis of matrix spaces, Fourier coefficients of automorphic forms, the Fourier integral and its applications, and Fourier integral operators and partial differential equations.

Dirichlet and Riemann series in the last century were inspired by Fourier series, and they eventually led to the *L*-series studied today. These ideas have unified number theory with the theory of group representations. Fourier analysis has led to the definition of function spaces (such as Sobolev spaces, Schwartz spaces, distribution spaces, and Hardy spaces) which form the basis of modern functional analysis. In this framework we can analyze differential equations (both linear and nonlinear) and their modern generalization—pseudodifferential equations—and Fourier integral operators. One can study the nature and propagation of singularities by these methods.

Although Fourier realized that his method was important—so important that he persevered for two decades in the face of intense criticism—he never realized just how fruitful his invention would be. While not every new development in mathematics has had the spectacular influence of Fourier analysis, the basic pattern has been much the same: the impact of good mathematical ideas spreads far, and in unexpected directions.

2. Computation

Perhaps the most striking mathematical application of this century has been the development of the electronic computer. It has become a fixture in offices, schools and factories and is fast becoming commonplace in the home as well. The point of this section is to show why the computer revolution is not simply an engineering revolution. It is a mathematical revolution; for the ideas central to the invention and everyday use of the computer are sophisticated mathematics.

Computers suffer from two fundamental limitations. Although the fastest computers can execute millions of operations in one second, they are always too slow. This may seem like a paradox, but the heart of the matter is: the bigger and better computers become, the larger are the problems scientists and engineers want to solve. The reach exceeds the grasp. When you double the amount of data in a problem, the number of steps needed to compute the solution often increases four-fold, or eight-fold, or sixteen-fold. In most applications, computing time is the most serious limitation on the frontiers of the possible. Doubling computer speeds every few years only means increasing their ability to solve larger problems by 10%–20%, often not even that. It takes an entirely new mathematical approach to enable a computer to get close to doing the needed arithmetic.

The second limitation of computers stems from their digital nature, since much of the mathematics that underlies science is continuous. Approximating the solution to a continuous problem with a discrete machine takes great skill. Most scientific calculations depend on the answers to such questions as: what mathematical methods lie behind a proposed numerical solution? Will errors produced because the computer deals with numbers of a fixed length (rounding-off errors) compound themselves to swamp the answer? Will other approximations yield such errors? If not, how long does a digital computation take to obtain a desired degree of accuracy? Again the mathematician must continually develop new points of view in order to improve the way that the computer handles a computation.

Mathematics is thus at the very heart of computation. But let us tell the story of mathematics and the computer from the beginning.

The computer itself. Nowadays it is easy to forget that the notion of an all-purpose computer is a very recent one. Until the last fifty years, a computing machine meant a machine designed to perform some particular piece of arithmetic. The Arabic mathematician Al-Kashi built one in the 15th century for computing lunar eclipses, and he built another for figuring the position of the planets. William Schikarf, Blaise Pascal, and Wilhelm Leibniz all fabricated machines for automatic addition and subtraction; Charles Babbage was famous for his Analytical Engine. To compute the area under a plane curve, J. H. Hermann, James Clerk Maxwell and James Thompson each devised planimeters, another sort of analog computer. However, the notion of a single, universal machine suitable for all problems and all calculations came from seemingly the least likely source—abstruse mathematical logic.

Logic and the computer. The foundations of mathematics rest on the foundations of logic. For centuries, mathematicians believed that deductive reasoning could never lead to inconsistent results. This conventional wisdom was called into doubt in 1903 by the famous paradoxes of Bertrand Russell and Alfred North Whitehead. For example, let S be the set of all sets which do not contain themselves. Does S contain itself?

About 1915, David Hilbert also embarked on a program to restore the foundations of mathematics. In 1927, John von Neumann, a young co-worker of Hilbert, published a famous paper conjecturing that mathematical logic would soon be proved free from possible contradiction. Yet only three years later, Kurt Gödel proved that even simple arithmetic contains "undecidable propositions," sentences whose truth or falsity cannot be proven. His method also demonstrates that a proof of the logical consistency of mathematics is impossible. The answers to this seemingly esoteric question turned out to have tremendous practical ramifications.

In 1936, Alan Turing and Emil Post realized independently that this question is equivalent to asking which sorts of sequences of 1's and 0's can be recognized by an abstract machine with a finite set of instructions; they envisaged such an automaton as a simple black box with a single long tape for writing and reading a single symbol. Turing and Post proved a surprising theorem about automata: in principle, there must exist a "universal automaton" capable of recognizing any sequence recognizable by any other automaton. That is, this universal machine could—with a finite sequence of instructions—imitate any other special purpose machine.

This was really the birth of the universal computer. The logical ideas were pursued further by Church, Kleene and others. But it was the great mathematician John von Neumann who realized how to implement the universal automaton as an electronic computer with stored instructions—a "program" which the machine itself could alter in the course of calculation. Von Neumann and his colleagues then undertook the monumental technical task necessary to make the theoretical a reality. Within a decade, devices like von Neumann's ENIAC, built at the Institute for Advanced Study in Princeton, were operating. At no point in the early years of the century would anyone have guessed where the esoteric debate on the foundations of mathematical logic would eventually lead.

Algorithms and computational complexity. We have already alluded to one of the central problems of computing: as the size of computational problems grows, the time and space needed to solve them grows far more rapidly. In the earliest days of computing, a mathematician would check the correctness and speed of a program by testing it on various different inputs, noting the time and space required. The drawbacks to this rough-and-ready method are obvious. To avoid them, mathematicians built upon the work of Turing and Post in devising theoretical models of computation to test how many operations an algorithm required. They realized that rather than trying to solve each new problem with specific, ad hoc tricks, it would be better to devise a collection of basic mathematical methods which could act as the building blocks for many algorithms.

Consider, as an example, a common task which a computer might perform many times: given n numbers $a_1, a_2, \cdots, a_n$, write them in ascending order. The simplest procedure is

(1) Write down a_1.
(2) Check if a_2 is less than a_1; if so, write it to the left of a_1. Otherwise, write a_2 to the right of a_1.
(3) Check if a_3 is less than the smallest number written down so far. If so, write it to the left. Otherwise compare it with the next number, writing it on the left of this number if it is smaller and on the right if it is bigger.
(4) Continue for $a_4, a_5, \cdots, a_n$.

A good measure of the time it takes to perform this algorithm is the number of comparisons which must be done. It might take as many as $\frac{1}{2}n^2$ if the numbers had originally been in descending order. This is called the worst-case complexity of the algorithm. Starting from a randomly chosen order, one would expect about $\frac{1}{4}n^2$ comparisons to be necessary, which is the average-case complexity. That the time needed grows with n^2 is the real limitation on how large a list can be effectively sorted. The particular coefficient $\frac{1}{2}$ or $\frac{1}{4}$ is usually neglected and computer scientists write that this algorithm takes time $O(n^2)$ to indicate that the time is on the order of n^2 steps.

We should remark that worst-case and average-case complexity of an algorithm can differ markedly. The well-known simplex algorithm for linear programming can require time exponential in the size of the problems in the worst case. However, these worst cases are few and far between; Borgwardt and Smale proved in 1982 for variants of this problem that, on average, the algorithm requires only time quadratic or linear in the size.

In fact, there is a much faster way to sort numbers, based on the principle of recursion: divide the numbers into two equal groups; sort each group; then combine these two sorted lists of $\frac{1}{2}n$ numbers. To sort each of the groups of $\frac{1}{2}n$ numbers, use the same procedure: sort them into two groups of $\frac{1}{4}n$ numbers, sort them and merge the lists. Sort each of the groups of size $\frac{1}{4}n$ by dividing it into groups of size $\frac{1}{8}n$ and so on. This recursive process takes time $O(n \log n)$, and so runs 64 times faster than the previous method when sorting 256 numbers.

The principle of recursion applies to many other problems as well. Computers are constantly multiplying large matrices—for example, in performing statistical data analysis. The standard high school method for multiplying $n \times n$ matrices requires time $O(n^3)$. However, there is a trick for multiplying 2×2 matrices which takes only 7 multiplications instead of 8. By recursively breaking large matrices into smaller and smaller pieces, this advantage applies to all the little problems and leads to a time $O(n^{2.83})$ algorithm for matrix multiplication.

In addition to recursion, there are many other useful methods to organize computation, such as "data structures." For instance, all the examples we mentioned in the introduction to this article—the Kalman–Bucy filter, image sharpening, CT scanners, linear programming and medical statistics—depend on using a computer to solve a system of n linear equations in n variables. Consequently, a great deal of attention has been given to such algorithms. The classical method of Gaussian elimination requires time $O(n^3)$. However, in many important problems—most prominently the finite element methods for solving differential equations or certain eigenvalue problems—necessary in computer simulations of weather, space flight, industrial design, etc., the coefficients in the equations include many zeros, distributed in a regular pattern. Mathematicians have exploited this structure to obtain a faster algorithm. The pattern can be turned into a graph (a structure consisting of points and edges connecting them), and the graph can in turn be used to devise a very efficient order for performing Gaussian elimination. The result is an algorithm that needs only time $O(n^{3/2})$—a major savings in a problem that must be performed thousands of times. Recently, it was shown that the method also generalizes to yield a time $O(n^{3/2})$ algorithm for matrices whose graphs can be drawn in a plane.

While many algorithms rely on rather elementary mathematical concepts, three recent algorithms exploit deep theorems from very different branches of mathematics to crack difficult computational problems. Some rather esoteric results have turned out to have quite practical computational consequences in testing for primes, graph recognition, and integer programming.

Large prime numbers form the basis for a new encryption scheme. But until recently, testing whether or not a 60-digit number is prime has been beyond the scope of even the fastest computer. The most straightforward test—checking all integers up to the square root of the number in question to see whether they are divisors—requires checking 10^{30} numbers. (Here 10^{30} is scientific notation for 1 followed by thirty 0's.) Number theorists, however, have long been studying the properties of prime numbers. Many of the laws they discovered—such as the so-called higher reciprocity laws—have recently been combined in a new algorithm for primality testing which makes it practical to check even 100-digit numbers.

Another frequent problem in computing is to decide whether two seemingly different n-point graphs are in fact isomorphic—that is, have the same pattern of connections. Until recently, we could only determine this through trial and error. However, a series of algorithms aims to discover symmetries between the two graphs, using a recent triumph of algebra, the classification of finite simple groups. The new algorithms based on symmetry run much faster than trial and error, although still better ways are being sought.

Another computer problem important in industry, integer programming, allows a company to optimize scheduling or use of materials. In the past few years, 19th-century techniques for studying lattices in algebraic number fields have been applied to integer programming and have led to new algorithms. (In addition, these same lattice techniques have provided the fastest algorithms for factoring polynomials.)

Aside from seeking faster algorithms for solving particular problems, mathematicians have begun to ask deep questions such as, "What are the absolute lower limits to how fast a problem may be solved?" and "Are some problems inherently intractable?" By building models of computation based upon Turing machines, they have begun to obtain preliminary answers. One topic of interest shows that certain questions, called NP-complete problems, cannot be solved in a polynomial amount of time. It may seem a rather negative aim to work to prove that a class of problems will remain computationally intractable. But such a proof would elucidate precisely what makes a calculation tractable, making it easier to find algorithms for the tractable problems.

Randomness in calculation. One of the most stunning mathematical discoveries about computation is that relying on chance—playing the odds, so to speak—can be far more effective than following any known predetermined algorithm.

The classic example is the Monte Carlo method, developed in the 1940s. For instance, to compute the area of a dart board mounted on a 100 square foot wall, throw 500 darts at random toward the wall. Assuming 15 darts land on the board, its area is roughly 15/500 the area of the wall, or 3 square feet. More generally, to compute the volume of a region R inside a box B, pick n points at random in B. A good estimate for the ratio of the volume of R to that of B is the fraction of the n points which lie in R. In fact the error in the method will tend to zero as more points are taken, and the rate of convergence is proportional to $n^{-1/2}$.

For complicated shapes and/or high dimensions the Monte Carlo method is extremely efficient. It has become a conventional numerical method to evaluate multidimensional integrals and standardizes the integration of functions that would otherwise be impossible. Even with the Monte Carlo method, some desired calculations by far exceed the capabilities of existing computers. Computer architects are investigating how to effectively link many parallel processing units, either for a general purpose computer or dedicated to one particular calculation.

Randomness in algorithms. The examples above illustrate an application of randomness in a continuum setting. Recently randomness has also proven very useful in studying algebraic problems. Here a random method yields exactly the right answer—except occasionally, when it gives the wrong one. Every such method depends on the fundamental structural properties of abstract algebraic objects such as polynomial rings, number fields, and permutation groups.

How would such an algorithm be useful? An algorithm that is correct 100% of the time might require n^6 steps to execute, whereas an algorithm that produced the right answer only 99.9999999999% of the time might need a mere n steps to perform. The tremendous savings in time would more than offset the small chance of error. In the last eight years, dozens of examples of such methods have been found.

One randomizing algorithm which will vastly improve computer security at a tiny cost will soon be hard-wired into silicon chips. This algorithm makes it possible to "fingerprint" a computer file in order to prevent anyone from tampering with it. Suppose that a computer has 40 megabytes of data stored on a disk. Consider the bytes as the coefficients of a 40,000,000th degree polynomial. Divide this polynomial by a randomly chosen 13th degree polynomial and save the remainder of the division. Now write down the coefficients of your randomly chosen polynomial and of the remainder and put them in your wallet. They are a "fingerprint" for the file. The chance that an interloper could change the data without altering this random fingerprint is less than 1 in 100,000,000,000,000,000,000, or 10^{-20} in scientific notation.

Primality testing, a problem we mentioned above, turns out to be very easy—if a tiny chance of error is allowed. If an integer n is not prime, then at least $\frac{3}{4}$ of the numbers between 1 and n have a particular property S which can be checked very quickly. If n is prime, no such numbers have property S. The test is simple: pick 50 numbers and check whether they have property S. If any do, then n cannot be prime. If not, then n is almost certainly prime—for, if n were not prime, the chance of picking 50 numbers at random without property S would be at most about $(\frac{1}{4})^{50}$ (or 10^{-30}). If those odds are not good enough, try another 50 numbers; it takes only a second or two, and the chance of error goes down even faster.

Computer assisted proofs. Mathematicians have used the computer as a scientific laboratory to test ideas and to develop precise conjectures, based on numerical evidence. This has lately been fruitful in the study of maps of the interval and, more generally, in the field of dynamical systems. Similar use occurs in number theory, algebraic geometry, topology, complex analysis, and the study of quasi-periodic potentials. Patterns emerge from extremely accurate computation. In some cases, detailed calculations even indicate when a function may have cusps or discontinuities and hence provide the basis for mathematical conjectures.

However, a fascinating new possibility has recently arisen: in certain cases the computer can act as a partner to the mathematician, rather than as a laboratory, in establishing a traditional mathematical proof. The computer can perform either an algebraic, combinatoric, or analytic task. For the former, as in the analysis of the four-color problem, the computer checks that a certain finite number of cases of a combinatoric statement hold—by checking them one after another. The reduction of the theorem to the combinatoric statement remains the job of the mathematician.

The computer can also verify inequalities necessary for the proof of a theorem. This can be done with 100% accuracy. The mathematician might have the computer check whether a particular number x lies with certainty within the interval $[a, b]$. Establishing the inequality $x < y$ reduces to establishing that the upper bound for the x-interval is less than the bound for the y-interval. The interval arithmetic can be done with certainty by reducing each calculation to integer arithmetic.

This technique has been used with success in studying iterations of maps of the interval, and it is an important ingredient in the recent proof of the existence of a fixed point for iteration of a quadratic map. The computer can check a large number of estimates, each of which can be established by hand, but which, as a whole, pose a problem of scale.

Both these examples are a departure from tradition which may well become increasingly important. At this stage we do not predict that computers will replace the thinking mathematician in outlining the architecture of a proof. But in some cases they will surely provide an aid to the mathematician which goes beyond the experimental laboratory or mathematical model: they will complement the mathematician's ability by establishing a large number of identities or inequalities within a mathematical proof.

Numerical analysis and mathematical modeling. Early numerical analysis can be traced to Newton (17th century) and Euler (18th century). However, discrete mathematics developed rapidly with the advent of the computer. In the period after 1950, the inversion of sparse matrices and numerical integration were widely studied, as were methods to integrate ordinary and partial differential equations. These advances made possible engineering design in a broad range of problems. Today most advanced technological design and development—from cars and aircraft to petroleum engineering and satellites—is based on computer simulation. In addition, real-time calculations led to the triumphs of space exploration and automatic rocket control.

Computations, however, do not run on automatic pilot. The importance of the intelligent mind cannot be overestimated. At one side of numerical modeling lie the relevant physical laws and mathematical equations which describe a particular engineering process. At the other side are numerical algorithms and codes (programs) used to instruct the computer. Connecting these two domains involves the mathematics of discrete approximations, and a mathematical understanding of the structure of the equations and of the nonlinear phenomena which they describe. Here finite difference and finite element methods have played a central role. With the advent of vector and parallel computing, problems long thought inaccessible are becoming more tractable. Despite this recent progress, numerical analysis of nonlinear partial differential equations in three dimensions, as well as many other frontier questions, still await new mathematical methods.

New mathematics can also make the difference in two important questions of speed. Can calculations be performed fast enough to be useful? In many engineering questions, an overnight turn-around is essential. Secondly, new mathematics can resolve whether a calculation is only feasible, or whether it can be performed in the real time necessary for a practical purpose like landing an aircraft.

Numerical mathematics plays a crucial role in three remarkable developments: the replacement of experiments by computer simulation, decision science, and signal and data processing. Computing can be cheaper than experimenting. It is much easier to modify experimental design in a computer study than in an actual physical experiment. In some projects experimentation is dangerous or impossible.

In aerodynamics, the design of aircraft, of turbines and of compressors is done with computer assistance. The ability to calculate aerodynamic forces on the space shuttle was an absolute necessity for the operation of the flight simulator in which the pilots of the space shuttle were trained.

Some other applications of numerical fluid dynamics are: the design of naval vessels, the calculation of combustion patterns, the flow of a mixture of oil and water (or other chemicals) in enhanced oil recovery, multiphase flow in reactors under transient conditions, the flow of ground water through crushed rock, and the propagation of sound signals through geological layers. Nuclear fusion reactor design relies on mathematical modeling: the plasmas at the densities that we wish to create are not yet available on earth; they exist only as mathematical models. The same is true of the development of laser fusion.

Operations research, an area of decision science relying on mathematical manipulation of stored data, has helped enormously to streamline large scale operations and insure the optimal use of resources. From its first industrial applications to the scheduling of petroleum refineries in the early 1950s, linear programming has resulted in substantial gains in the efficiency of the operations it was used to analyze. Topology, convex analysis, combinatorics and geometry all contribute to the mathematical model.

Digitally stored information can be manipulated mathematically to extract particular details hidden in a mass of data. Here the mathematics of statistics enters to point up hidden trends and correlations. In medical science, physiological modeling opens up possibilities for understanding the basic biological functions of organs as well as the design of effective prosthetic devices. Perhaps the most spectacular success of medical computing is its use in computerized X-ray tomography, as well as other advances in noninvasive diagnosis which make use of ultrasonics, nuclear magnetic resonance and positron emission tomography.

The full impact of the computer will develop over coming decades. As with the invention of the steam engine, the computer has an enormous potential to enhance our lives. Initiatives currently being discussed for supercomputers and fifth generation computers (aimed at artificial intelligence, pattern recognition, knowledge processing, etc., as well as scientific computing) can only partially succeed unless such hardware is complemented by new mathematical points of view.

The architecture of the present computer is sequential; it is called the von Neumann machine after the man who showed how to implement this idea of programming. In the coming revolution of computing hardware, organization will be parallel rather than sequential. It will be designed much more like the slower but far more complex parallel structure of the brain and will certainly require a new conceptual approach to harness and to program. For all its uses, the computer depends critically on ideas, insights and methods from mathematics.

3. Mathematical Physics

Modern theoretical physics provides the most striking example of the way mathematics and science complement one another. Whole fields of mathematics have been born from attempts to understand the laws of physics. Reciprocally mathematics has provided the language of physics. In overview one sees parallel developments in these two subjects. We describe here a few of the triumphs of mathematical physics.

Group theory and quantum mechanics. The theory of group representations has had far-reaching impact on quantum theory and more generally on understanding the symmetries of nature. An object is said to have a symmetry if it is essentially unchanged by some transformation. An equilateral triangle, for example, is unchanged if it is rotated about its center by 120 degrees. Group theory is the study of transformations preserving an object, rather than the object itself.

In the late 18th century, Lagrange observed the significance of (the group of) root-interchanges for understanding third and fourth degree equations. However, many years passed before 1832, when Galois perceived the importance of studying the general structure of root interchanges for all polynominal equations. The discovery of the Galois group is generally taken as the birth of group theory.

Once the abstract notion of a group had been formalized, several major developments occurred. Felix Klein recognized that groups were a natural tool in the study of geometric symmetry. At about the same time, Sophus Lie discovered the connection between groups and the theory of differential equations. Meanwhile the theory of group representations began with work of Frobenius, E. Cartan and others who related the abstract theory of groups to concrete matrix groups; the matrices were said to "represent" the group.

Remember, however, that over the period of 100 years during which the theory of groups was invented and developed, physicists and experimental scientists virtually ignored it. Yet the theory of groups was marvelously suited to physics, both to classical physics and even more centrally to quantum theory.

In classical mechanics Noether developed the general connection between symmetry groups and conservation laws of classical mechanics, such as conservation of energy or conservation of angular momentum. In quantum theory the developments were even more striking. Only a short time passed between the discovery of electron spin in 1925 and the work of E. Wigner and H. Weyl to interpret spin as an aspect of group theory. The question was how to explain the existence of two states of the electron—spin up and spin down—which in turn explained the splitting of spectral lines of light occurring when one placed light-emitting atoms into a magnetic field. The answer lay in understanding a group called $SU(2)$, which is related naturally to the group of rotational symmetries of the three-dimensional space in which we live. The two spin states of the electron are interpreted as elements of a "fundamental representation" of $SU(2)$.

After that discovery, group theory became a central tool in physics and chemistry. For instance, the classification of emission and absorption spectra from atoms and molecules (including the qualitative and quantitative analysis of atomic and molecular spectroscopy) boiled down to the study of representations of the permutation group and the group $SU(2)$. The Coulomb potential is unchanged by rotations in three-dimensional space. The group representations precisely describe how a physical state fails to share the full rotational symmetry of the underlying force. Finite dimensional subgroups of rotational symmetries (point groups) describe crystal symmetries and many other symmetries of condensed matter physics and chemistry.

In 1926 Pauli discovered another symmetry of the Coulomb force, different from the rotational symmetry discussed above. This symmetry arose from an understanding of Lie's theory applied to the "eccentricity" of a classical elliptical orbit. (This eccentricity invariant of motion, studied by Laplace, Runge, and Lenz, describes how much the ellipse differs from a circle.) The properties of this extra symmetry led Pauli to a simple, elegant picture of the quantum mechanical hydrogen atom. More important, this work foreshadowed the idea of studying space-time symmetry in conjunction with other types of symmetry.

A great step forward came in 1939 when Wigner analyzed the positive energy representations of special relativity, i.e. representations of the group of Einstein, Lorentz and Poincaré. The mathematical tools of the day required generalization to solve this problem, and Wigner built on the classical work of Frobenius. One consequence of his analysis is that every representation of the relativity group was characterized by two intrinsic numbers, its "mass" and its "spin." In this way, both mass and spin derive from a fundamental symmetry, namely special relativity. After that discovery, a physical particle in quantum theory could be interpreted as a mathematical object, namely as a group representation.

Staggering development ensued in the mathematical theory of Lie groups and their representations. In turn, these purely mathematical developments came to play a central role in modern number theory, geometry, ergodic theory, etc. This work is ongoing, not only for Lie groups but also for infinite dimensional groups, such as groups of diffeomorphisms, the Weyl group, and gauge groups.

Such a theory of representations later played a role in revolutionizing ideas in physics. At first, group theory was applied to describe laws of nature. Later a modern point of view evolved in which groups actually became part of the statement of the laws. It was not long before physicists accepted as fundamental certain symmetries of nature other than space-time symmetries. Nuclear forces do not depend on the electric charges of the particles involved. Heisenberg described this independence as the symmetry of the nuclear force under a symmetry which relates protons and neutrons. He called this "isotopic spin" and regarded the proton and the neutron as two different states of one particle, the nucleon. These two states are characterized by a two-dimensional representation of $SU(2)$.

Not only has this point of view become standard, it has been considerably extended. Physicists believed that the fundamental laws of nature were invariant under certain reflection symmetries. The mirror reflection symmetry in three-dimensional space had been taken for granted until the late fifties, when Lee and Yang pointed out that it should be tested. They proposed experiments which demonstrated that parity was not an exact symmetry of nature! After this breakthrough, the question of which apparent discrete symmetries are found in nature has been in the forefront of physics.

Why would an expected symmetry not occur in nature? Although symmetric equations may have a simple structure, the symmetry may not apply to nature. On the other hand there is a more elegant possibility: the laws themselves have a symmetry, but the particular solution of interest does not possess it. For example, Newton's gravitational potential is rotationally symmetric, but a classical planetary orbit is not necessarily circular—it can be elliptical or hyperbolic. One expects, however, that the lowest energy state of a system would possess all the symmetry of the laws themselves. If it does not, then the symmetry is said to be "broken."

Broken symmetry appeared first in the physics of magnetization and in the chemistry of phase transitions (such as boiling or freezing of water). Broken symmetry occurs in a model introduced by Lenz and Ising and studied in famous work of Peierls in 1936 and Onsager in 1944. It has become the standard mechanism to describe many effects in statistical physics. Physicists do not know whether parity violation can be described as broken symmetry, but in fact broken symmetry has achieved a crucial role in the description of other aspects of particles.

Starting in the 1950s large accelerators produced dozens of new particles. Soon it became necessary, or at least very desirable, to explain them coherently. How could one explain the masses, the spins, and the other intrinsic properties of these particles, as well as their interactions with one another? The physicists again turned to symmetry to simplify the problem.

Mathematicians for reasons internal to their field had been developing the abstract theory of representations of compact groups. As it turned out, this understanding of group representations provided exactly the information required in the search for the laws of nature. In 1961, Gell-Mann and Neeman proposed extending the proton-neutron symmetry of Heisenberg to the larger group $SU(3)$. Besides explaining the charge independence of nuclear force and neatly classifying the many new particles by their properties, they added a startling new observation. The representations corresponding to the familiar observable particles (protons, neutrons, mesons, etc.) could all be constructed from products of two fundamental representations. Each component of the fundamental representations was dubbed a "quark."

This mathematical picture of fundamental particles composed of quarks predicted the existence and mass of a new particle called the omega, which also had to possess a quality called "strangeness." After the omega particle was found in 1964, the $SU(3)$ symmetry became accepted, as well as the notion that the unseen quark was a fundamental component of the laws of nature.

These twenty-five years made it clear that groups and their representations were as essential to modern particle physics as the more traditional tools of complex analysis and partial differential equations. It is tempting to speculate on the relation of some of the newest advances of group theory to physics. Two beautiful mathematical theories are the classification of finite groups (related to the recent discovery of a "monster" group), and supersymmetry. Only time will reveal whether nature embraces these mathematical notions in some subtle way.

Differential geometry and physics. The early history of differential geometry dates to the 17th-century work of Fermat on curves, and to Gauss who studied the curvature of surfaces in the 19th century. Gauss' point of view could be called the concrete approach, since he studied surfaces embedded in a Euclidean space of higher dimension. Riemann formulated the geometry of surfaces as entities in their own right in 1854. Eventually geometry incorporated the algebraic notions of symmetry and groups. What emerged was tensor analysis, a subject founded by Bianchi, Levi-Civita, Christoffel, Ricci, and others.

Einstein embraced this intellectual framework to explain his fundamental ideas about gravity, proposing his General Theory of Relativity in 1915. Einstein's basic equation sets the curvature of space proportional to the density of energy; the fundamental constant of proportionality is defined to be the gravitational constant. From this point of view, gravitational force results from the curvature of space. Relativity theory yields Newton's force law for gravitation as the limiting case of a space-time with small curvature.

The second fundamental force of classical physics is electromagnetism. In 1918 mathematician H. Weyl observed that the electromagnetic forces could necessarily be inferred from the geometry of space. He based his study on scale transformations of space; since he thought of measuring space by a "gauge," Weyl called electromagnetism a "gauge theory."

This conceptual advance was not fully appreciated at the time, but the gauge picture ultimately led to our modern effort to unify the four fundamental forces: gravity, electromagnetism, strong forces, and weak forces. What startled physicists nearly forty years later was a simple but profound generalization of electrodynamics (as described by the basic equations of Maxwell dating from 1873, reinterpreted by Weyl, and also incorporating the equations of Dirac). In 1954, Yang and Mills suggested that the basic symmetry group of electromagnetism be enlarged to include a group describing the symmetry of strong forces. They considered the simplest equations which were compatible with this invariance, and which reduced to Maxwell's equations for purely electromagnetic forces. Today this subject is known as "nonabelian gauge theory," since the basic symmetry group is a noncommutative group. Here the choice of the particular group of symmetries is crucial for physics; it is an explicit example of the philosophy that discovering the symmetry group is a part of finding the laws of nature.

The notion of a Yang–Mills gauge theory was not at all new. Some years earlier, mathematicians had introduced the global geometric notion of a fibre bundle and had recast Riemann's geometry into fibre-bundle theory. A fibre bundle is a space consisting of many similar spaces pasted together. For example, a torus (doughnut) can be assembled by pasting together successive circular cross sections. Mathematicians introduced the notion of a "connection" as an object to measure the local twisting due to the curvature of such a space. An enormous theory was laid out, including the study of the topology (global properties) of abstract spaces with curvature. Many algebraic and geometric invariants, such as Chern numbers, Stiefel–Whitney classes, index invariants of Atiyah, Singer, Hirzebruch, Weil, Bott, and others were discovered as part of the general theory. What the physicists had added to this picture was the notion to find such structures as solutions to a set of variational equations. These nonlinear differential equations which the connection satisifies are the natural generalization of Laplace's equation within the framework of differential geometry.

Let us take a brief technical excursion. In the original equations of Maxwell, the electric field $\mathbf{E}(x, t)$ and the magnetic field $\mathbf{B}(x, t)$ each are vectors—three component objects. Each component of these vectors is a real valued function on space-time. The modern view of a gauge theory is to consider the component functions $E_i(x, t)$ as elements of the Lie algebra of the gauge group. In the case of electromagnetism the group is $U(1)$ whose Lie algebra is just the real line, yielding ordinary real-valued functions $E_i(x, t)$ as in Maxwell's theory. The Yang and Mills generalization was to replace $U(1)$ by a larger compact matrix group whose Lie algebra consists of noncommuting matrices. Thus each component of the electric and magnetic field is a matrix, rather than a number. This matrix varies from point to point as one moves in space and time. While physicists are still not certain which group is the most fundamental one to choose, a typical candidate would be $SU(3)$ or $SU(2)$ or $U(1)$, which characterize, respectively, known symmetries of strong forces, weak forces, and electromagnetic forces.

Analysis and quantum fields. Physicists have believed for over sixty years that quantum theory provides the correct framework to describe fundamental, or submicroscopic, particles. Thus modern physics must find a mathematical theory that encompasses gauge theories, as well as quantum mechanics and special relativity. Such a combination is known as a quantum field theory. Other examples of field equations are the Dirac equation for the electron, nonlinear scalar wave equations, and the Einstein equations for gravitation.

The challenging and elusive search for the mathematical foundation of quantum physics has inspired new mathematics and succeeded in yielding important insights into physics. At present it provides an exciting opportunity to unify these fundamental sciences.

As early as the 1920s when modern quantum mechanics was born, many of the world's greatest mathematicians, such as Hilbert, von Neumann, and Weyl, felt strongly attracted to this new physics. The mathematics of wave propagation, of integral equations, of differential equations, of eigenvalue problems, of linear analysis, of probability theory and of group theory, all contributed to the understanding of nonrelativistic quantum theory. These areas of mathematics also profoundly influence every area of modern physics and engineering.

Let us now focus attention on a puzzle in mathematical physics which arose in the 1930s with the attempt to incorporate into quantum theory the effects which arise because particles can affect themselves, indirectly, through their effects on other particles. For example, a typical experiment might measure the frequency (color spectrum) of light emitted by an excited atom. In the process the light interacts with the atom, while the atom in turn interacts with the light. Because of this circle of effects, the light can affect itself, and the phenomenon is said to be nonlinear. However, every attempt to derive observable frequency shifts from this nonlinear system resulted in infinite answers!

In 1947, a conference was held on Shelter Island to focus attention on the major open problems in theoretical physics and to reorient the researchers in the postwar period. This conference became famous because it stimulated the formulation and application of a set of rules to carry out the mystifying calculations; these rules systematically ignored meaningless quantities such as infinity or division by zero, i.e. 1/0. Yet they yielded definite answers.

The attention of physicists was directed toward understanding a small, recently-observed correction to the spectrum of light emitted by hydrogen, today called the Lamb shift. A second effect of the nonlinear interaction concerned the magnitude of the energy of a single electron in a magnetic field. According to Dirac's theory, the value of this magnetic energy, or "magnetic moment," would exactly equal 2 in standard, dimensionless units. In fact experiments yielded the number 2.002, and the deviation .002 from 2 was dubbed the "anomalous" magnetic moment.

Over the last thirty-five years, countless man-years of labor have permitted the application of these rules to the first few terms of a power series in the electric charge. The computations involve the exact evaluation of thousands of integrals. The program is so immense that large computers were called upon just to carry out algebra. The result accurately predicts the magnetic moment of an electron. On the side of experiment, improvements over the years have yielded the present observational value of 2.002319304, one of the most precisely measured quantities in physics. The calculated number and the observation agree down to the last decimal place. Because of this extraordinary check between experiment and prediction, the rules underlying the calculation are taken seriously. In time, these rules for ignoring infinities became known in physics as the "theory of quantum electrodynamics."

Quantum theory's ability to explain the Lamb shift and the anomalous magnetic moment set the stage for a new era in quantum physics. The development of these ideas eventually led to the Yang–Mills theories described in the previous section. With twenty years' additional work, physicists came to choose the $SU(2) \times U(1)$ symmetry group to describe and unify two fundamental forces of nature: electromagnetic forces and weak forces. Glashow, Salam and Weinberg received the 1979 Nobel prize for this work. In a more speculative proposal, an $SU(5)$ gauge theory also unifies the strong forces. This "grand unified" theory predicts that the proton is unstable—i.e., that it will eventually decay. So far this phenomenon has not been observed in the large experiments currently searching for such a decay. In any case, quantum field theory has become the accepted basis for quantum physics.

However, the word "theory" is not used here in the traditional sense, at least not according to the standards of scientific explanation common in physics before the era of quantum fields. Because of the infinite (or otherwise ill-defined) quantities that physicists initially ignored in formulating their rules, we must ask whether this method actually has a mathematically consistent formulation. In other words, "Can relativity in combination with quantum theory be incorporated as part of traditional mathematics?"

The rules for dealing with the infinite quantities described above are known in physics as "renormalization." Today, some forty years later, this problem is only partially understood. But mathematical progress has helped make quantum field theory approachable and has led to a formulation we believe will succeed. This formulation involves generalization of both the differential calculus and the integral calculus to the case in which the unknown functions depend on an infinite number of variables.

In the usual calculus one differentiates and integrates functions $f(x)$, where x is a point in a finite dimensional space. The generalization is to consider the mathematics of functions $f(x)$ where the variable x has infinitely many coordinate directions. The subject in which one differentiates or integrates functions of an infinite number of coordinate directions is called the "functional calculus."

The differential functional calculus can be traced to the work of the famous Italian mathematician Vito Volterra, in his study early this century of general partial differential equations. Extensive developments of these ideas have continued ever since, and the field of functional analysis remains central for understanding physics.

In recent years the integral functional calculus has also played a major role. The original ideas in this field appeared in the theory of probability, as developed by Norbert Wiener. He abstracted the notion of integrals of functionals in an attempt to understand diffusion and Brownian motion. In this way, Wiener could represent the solution to the heat diffusion equation as an integral over classical particle trajectories. In physics a related point of view emerged from work of P. A. M. Dirac and R. Feynman in the 1940s, and today it is known in physics as the "sum over histories" approach to quantum theory. The connection between these two ideas has led to an understanding of how Wiener's integral fits into modern physics. It also opened up the mathematical development of "functional integrals," starting in the 1950s, by M. Kac, I. Gelfand, and many others.

The functional calculus eventually has been developed in recent times to the point that it could be used to tackle quantum field theory and the infinities of renormalization. In its modern form, this relatively new area of mathematics and physics has become known as "constructive field theory."

Reunification of mathematics with physics. The above discussion clearly points toward an exciting development taking place right now: reunification of mathematics with theoretical physics. After the advent of quantum theory in the 1920s, mathematicians and theoretical physicists began to move apart. Perhaps physicists believed that it was impossible to give a complete explanation in the traditional framework and still keep sight of the increasingly complicated set of physical phenomena. Mathematicians, on the other hand, found physics difficult to understand because the foundations were not treated properly, from their point of view. For whatever reason, each subject developed a special vocabulary, hard for the specialist in the other to penetrate. To make matters worse, study of one discipline by workers in the other was generally discouraged.

Past decades have seen internal unification revolutionize both these subjects. Mathematicians discovered deep relations between group theory, topology, algebraic geometry, differential geometry, analysis, and number theory. Meanwhile physicists discovered intimate connections between particle physics, condensed matter physics, and finally astrophysics. Twenty years ago a professor of mathematics and a professor of physics at the same university rarely had scientific contact. Today we sense excitement as the entire disciplines of mathematics and theoretical physics are coming together.

To illustrate this phenomenon, we here mention several examples of current work. Constructive field theory, developed by Glimm, Jaffe and others, has resolved much of the fifty-year-old mystery of the foundations of field theory. A new area of mathematics has been created that provides a general framework, dictated by physics, within which one can answer the questions. A complete theory, including renormalization, has been constructed for several quantum field examples. The main reason that the present answers are incomplete is that these examples simplify the presumed equations of physics.

From the point of view of the integral functional calculus, quantum field theory can be regarded as the study of a probability distribution for classical fields. A generalization of probability theory emerges, with many new and challenging mathematical aspects. Similar mathematical problems arise in classical statistical physics. The relation between these subjects also explains at a fundamental level why phenomena known in statistical physics—phase transitions and symmetry breaking—should appear in quantum physics.

The heat kernel $\exp(-tH)$ generates a random process labelled by time. Such random processes arise both in pure mathematics (geometry and topology as well as analysis) and in many fields of application, such as electrical engineering, stochastic control theory, and presumably econometrics and population biology. The random field (labeled by space as well as time) has the infinite dimensional Laplacian H as its generator. We suppose that its abstract theory and applications would be as rich as those of the random process indexed by time alone. For example, the functional integral methods—and related problems in stochastic differential equations—appear intimately related to representation theory for infinite dimensional groups, such as "loop groups."

Two ideas useful in mathematical study of renormalization are phase cell localization and the renormalization group; they have also had very fruitful applications to understanding phase transitions in physics. In mathematics, related notions appear in the theory of micro-local analysis and in harmonic analysis, such as Fefferman's study of the spectrum of the Laplacian using the Heisenberg "uncertainty principle." We expect that such relations will become clearer in time.

We have also mentioned the focus on geometric questions in physics. One recent puzzle in classical relativity theory was how to define the total "energy" of the universe. Physicists have proposed a definition of energy for spaces in which the curvature vanishes rapidly at infinity. The important positivity property of the energy (which in the quantum theory is also at the heart of the work on constructive field theory) was established two decades later, about 1980, by the geometers R. Schoen and S.-T. Yau. Their method to solve this physics problem developed the mathematical theory of minimal surfaces in a manner important to the ongoing study of singular harmonic maps and nonlinear differential equations.

In the last ten or fifteen years mathematicians and physicists realized that modern geometry is in fact the natural mathematical framework for gauge theory. The gauge potential of physics is the connection of mathematics. The gauge field is the mathematical curvature defined by the connection; certain "charges" in physics are the topological invariants studied by mathematicians.

While the mathematicians and physicists worked separately on similar ideas, they did not just duplicate each other's efforts. The mathematicians produced general, far-reaching theories and investigated their ramifications. Physicists worked out details of certain examples which turned out to describe nature beautifully and elegantly. When the two met again, the results were more powerful than either anticipated.

In mathematics, we now have a new motivation to use specific insights from the examples worked out by physicists. This signals a return to an ancient tradition. In physics, this understanding has focused attention on geometric questions. One aspect of the mathematical theory explains the observed quantization of magnetic flux in superconductors. Another well-studied example in gauge theory is the predicted existence of an elementary magnetic charge, or monopole, i.e. a magnet with a north or south pole, but not both. Some current experiments are searching for such a particle, but have not detected it.

In 1981, M. Freedman established the four-dimensional Poincaré conjecture, a problem unresolved for over sixty years. His theorem gives a partial topological classification of four-dimensional spaces. One suspected that Freedman's classification would also carry over when one required "smoothness" of the space, or a differentiable structure. At least intuition says so. In fact S. Donaldson's recent theorem says that this expectation is false. A physics interpretation of certain solutions to the Yang–Mills equations of classical gauge theory, followed up by a mathematical study of these equations, proved to be very important for the proof of this abstract theorem in mathematics. As a corollary, it turns out that a new, "exotic" Euclidean 4-space exists. Many topologists are now studying gauge theories; physicists are now studying topology. It appears that rich, new insights into the topology of four dimensions will result from this synthesis.

Another interesting development during the past few years is the study of "supersymmetry" algebras and the construction of supermanifolds. Mathematicians have known superalgebras as graded algebras, and one way to realize supersymmetry is using the standard De Rham complex. Physics introduced a new feature by constructing supergroups, supermanifolds and superfields associated with these algebras. The Laplace operator (Hamiltonian) can be represented as a function of superfields. The Atiyah–Singer index theorem—a highlight of modern mathematics which unified ideas in topology, geometry and analysis—can be proved using this method. Supersymmetry gives a new point of view on the index theorem and links it to the interactions (Lagrangians) of modern physics.

It is no surprise that this discovery has captured the imagination of both geometers and theoretical physicists. A related discovery is that "anomalies" of quantum physics—classical equations which fail in quantum theory, and related to defining the determinant of the Dirac operator—can be viewed as an aspect of K-theory, an abstract machine in modern topology and geometry. In fact K-theory even appears to be related to the spectrum of Schrödinger operators with quasi-periodic potentials. Such equations arise in describing magnetic properties of materials with random defects.

We are just scratching the surface of a new set of ideas whose natural setting embodies both mathematics and physics. We appear to be entering a new era where the boundaries between mathematics and theoretical physics practically disappear.

4. Communication

As high-speed electronic communication becomes commonplace, there is a tremendous need for better transmission schemes—ones that minimize the effect of inevitable transmission errors, ones that protect confidential or secret messages, ones that route messages most efficiently. Many of the best schemes are based on patterns or properties of classical algebraic and geometric objects, originally studied for their intrinsic interest. Mathematically, these are the subjects of information theory, coding and encryption.

Coding theory: Protecting against errors. Consider the difficult task faced by a Mariner spacecraft sending back to Earth intricate images of the Martian surface. The messages it beams back will necessarily be garbled by random noise and, unless some amount of redundancy is built into the messages, scientists at NASA won't know whether the data that they receive are correct. One solution might be to repeat the message, say, five times, allowing the receiver to compare all the versions and make a good guess as to what was intended. This procedure, however, is very wasteful; the spacecraft can transmit at only one-fifth the rate, and soon its memory will overflow with pictures it has taken but not yet transmitted. Closer to home, the same problem arises with static on a telephone line or even with random errors in stored data, such as bank account balances.

In the earliest days of high-speed communications, the task of building in redundancy without too great a loss in transmission rate was very much a hit-or-miss procedure. Soon, however, mathematicians realized that the question could be approached systematically. First, information theory and probability could be used to study the problem of determining what message was likely to have been sent. Second, the codewords in a coding scheme could be chosen to correspond with the elements of some algebraic or combinatorial object (like a vector space or a graph); the mathematical properties of these objects then could be used to estimate the error-correcting power and transmission rate of the code and thus to find efficient codes. Some of the most common algebraic codes today, for example, use properties of the geometry of lattices in n-dimensional space and the automorphic forms associated with them, or finite geometries and their symmetry groups, or the behavior of the roots of polynomials over finite fields.

A stunning example is Goppa's recent suggestion of a novel way to use algebraic geometry to generate codes. (Goppa is a distinguished Soviet expert in the theory of codes.) Specifically, he started with a curve X over a finite field, certain distinguished points $p_0, p_1, \cdots, p_n$ on X, and meromorphic functions $f_1, \cdots, f_n$, where f_i has a simple pole at p_i and possibly a pole at p_0. The allowable messages, or codewords in the scheme, would be those n-tuples $(c_1, \cdots, c_n)$ with the restriction that $\sum c_i f_i$ has a zero of order w at p_0, for some w chosen in advance.

The point of such a complicated construction is that algebraic geometers have long studied these objects. The famous Riemann–Roch theorem provides an estimate of the transmission rate of such a code. Similarly the error-correcting power of such a code can be determined by estimating the number of zeros of such curves of a given genus. This has also been a topic of great interest in the algebraic geometry research of Deligne, Rapoport, Ihara, Langlands, and others.

Recently Tsfasman, Vladut, and Zink have applied Goppa's method by using Shimura curves with supersingular points, objects long cherished by mathematicians studying number theory, group representation theory, automorphic forms and algebraic geometry. Some of the codes obtained not only are better than the best previously known, they are better than the Gilbert–Varshamov bound (a particular bound on efficiency which had been assumed by many to be the limit of how efficient a code could be).

We do not know how practical these new codes will be to implement, but their discovery illustrates how coding theorists find unexpected applications of other, often esoteric branches of mathematics. The flow goes in both directions, however; the sort of questions that coding theorists ask about geometry are at times different from those geometers have studied. Geometers estimate zeros of curves of varying genus over a fixed field; coding theorists fix the genus and vary the field. In this case known methods in geometry could be generalized to yield the desired coding bounds.

Encryption: Sending secret messages. Encryption is the process of scrambling a message to make decoding impossible. It has been a hot topic of mathematical interest since 1976 when Diffie and Hellman proposed the idea of a public-key crypto system (PKC).

Such a system exploits mathematical "trap-door" operations, i.e. functions much easier to evaluate than to invert. For example, it is much easier to add together a collection of numbers chosen from a set than to inspect the sum to figure out which were the numbers added. Merkle and Hellman used this notion to create the first PKC. Rivest, Shamir, and Adleman created another scheme based on the fact that multiplying two prime numbers together is simple, while determining what the factors were from the product is very difficult. This scheme has received wide attention.

What makes PKCs unique is that the sender and receiver never need to exchange the secret key for the cipher. For example, in the second scheme above a recipient would announce a "public key" consisting of a large number N and an integer r. Anyone wishing to send a message to this individual would scramble his message according to a simple procedure: consider the digital message as an integer modulo N (breaking it into blocks if necessary) and raise this integer to the rth power modulo N. There is a second integer s such that raising the encrypted message to the sth power unscrambles it. The catch is that the only known way to compute s requires knowing not just N, but the prime factors of N as well—information which the recipient keeps to himself. So, the recipient has a way of decrypting, but anyone else must first factor N.

Factoring an integer N is a surprisingly hard problem; the best known algorithms take a long time. The most straightforward procedure may require testing up to $N^{1/2}$ numbers as potential divisors. Better methods have been devised which take $O(N^{1/4})$ steps; in fact, the number of steps can be brought down to $c(\varepsilon)N^{\varepsilon}$ for any $\varepsilon > 0$. Here $c(\varepsilon)$ is a constant depending on ε, which is believed to grow very rapidly for $\varepsilon < 1/4$. These algorithms, however, are far too slow to factor a 100-digit number.

By contrast, since primality testing can be carried out quickly, a recipient in a public key system can easily choose a 100-digit number N by finding two 50-digit primes and multiplying them together.

Whether such an encryption scheme is secure enough for important government and commercial communication depends on how hard prime factorization really is. If factorization were known to be an essentially intractable problem, the novel and convenient scheme could be used with full confidence. On the other hand, if a very fast algorithm were known, it would have to be abandoned entirely. And so the issue of security hangs upon questions which a decade ago would have been thought of esoteric interest but of little practical importance.

As PKCs become more widely used, mathematicians will face the increasing challenges of attempting to crack them. Already, the original scheme of Merkle and Hellman has been broken by Shamir, who showed in 1982 that integer programming techniques can detect patterns in the scrambled messages, making unauthorized deciphering possible. By contrast, many researchers believe that prime factorizaton is an essentially hard problem. Still, if we use the history of mathematics as a guide, a revolutionary method of factoring should not be discounted. All we can say for certain is that in the next decades some very pure mathematics will take on some new and important ramifications.

5. Engineering

Engineering provides an excellent model of the interaction between mathematics and the other sciences. We include here those areas of classical physics concerned with the gross behavior of matter, including the mechanics of solids, fluids, electromagnetism, chemical reactions, etc. Much of the mathematics which arises is nonlinear, and for this reason the questions are especially difficult and challenging. The overall subject is so diverse that we can discuss only a few selected topics in the sections which follow. A recurrent theme is the interplay between asymptotic and numerical analyses. The isolation of leading contributions may require formulation of new mathematical models. Numerical methods require new mathematics as well. Again, we are not attempting to be representative, but rather to provide generic examples.

Differential equations. One of the most active branches of mathematics is the theory of differential equations. As discussed in the section on Fourier, harmonic analysis led to the classical understanding of heat and light through the study of the diffusion equation and of Maxwell's equations. These are only two examples of linear differential equations central to engineering. The general methods have been highly developed in the case of linear equations. Here detailed information has been established on properties of solutions to equations which govern our every movement. An understanding of characteristics has been essential for engineering insight into wave propagation and fluid flow. Fourier analysis and its generalizations are such standard points of view that one almost takes linear differential equations for granted; yet they are the basis for a huge fraction of mathematics. Linear equations also lie at the foundation of nonrelativistic quantum theory and hence at the understanding of materials.

Nonlinear differential equations date to the time of Newton and his study of the planets. These equations tend to be harder to understand, especially nonlinear partial differential equations. Fewer solutions are known in closed form. (Special solutions to special equations with an interpretation in nature, such as solitons, have achieved widespread use in engineering and physics models.) Furthermore, the methods to understand one equation seem maddeningly inappropriate for another! However, it is these equations which are important in describing the chemical reactions in a combustion engine, fluid flow under most conditions, magnetohydrodynamics, or stresses in solid bodies. Generically, the equations that describe extreme temperatures, forces, or pressures tend to be nonlinear. Hence many of the most important engineering problems center on the understanding of nonlinear effects. Clearly a theoretical understanding of the equations is important both for qualitative and quantitative questions of design.

Nonlinear equations can also have more than one solution for a given set of boundary values or initial conditions. The question of whether and when this happens for a particular equation is the subject of much current research. The bifurcation process which can occur with the onset of nonuniqueness clearly is important for structural stability, chemical processes, and turbulent flow. We touch on many other aspects of differential equations throughout this section.

Complex function theory. Complex numbers were introduced in the 16th century to solve quadratic equations. Only some 300 years later did Gauss demonstrate that the roots of every algebraic equation are complex numbers. The theory of functions of a complex variable emerged as a fundamental area of mathematical research due to his influence and that of Cauchy. The famous Cauchy integral theorem was proved in 1825; Cauchy also laid the foundations for the theory of elliptic integrals. Twenty-five years later, Riemann vastly enriched the subject, discovering connections between problems in physics on the one hand, and those in complex function theory on the other. Riemann's results and conjectures inspired a whole succession of further developments, including the unification and clarification of the integral transforms we now associate with the names Fourier, Laplace, Poisson, and Hilbert.

Complex analysis has permeated engineering. A major reason behind the success of this method is that by using complex numbers two-dimensional problems can be handled the way one-dimensional ones had been previously. While vectors also simplify multi-dimensional analysis, they obey a different calculus from numbers. Using complex numbers, one can either study problems depending on two variables (for example three-dimensional problems with a symmetry) or problems involving two real-valued functions which could be treated simultaneously as one complex-valued function.

By 1920, scientists at Bell Laboratories were making systematic use of complex function theory in the design of the filters and high gain amplifiers which made long distance telephone communication possible. A notable example of the importance of complex function theory is the Nyquist criterion for the stability of feedback amplifiers—an aspect of the "argument principle" in complex analysis. While mathematically straightforward, the Nyquist diagram became a marvelous tool for understanding and defeating feedback instability; it is now taught to every engineer.

Conformal mapping techniques have been used to solve a host of problems along the lines envisioned by Riemann. As might be expected, the general applicability of complex analysis to two-dimensional problems became legend. For example, Joukowski used complex mapping techniques to specify the shape of an airfoil and to analyze the flow pattern around it, revolutionizing airplane design. Complex function theory became a central tool in the description of fluid flow, and in the design of cars and ships.

Time series and control theory. Norbert Wiener's scientific career represents an unusual achievement in mathematics, because much of his most abstract and theoretical work had "instant" applicability. His theory of time series analysis, which he developed during World War II to aim artillery, became a focal point of modern control theory. In fact, the original version of his classic paper, "Extrapolation, Interpolation and Smoothing of Stationary Time Series," was a classified document. Because of the color of its cover and the impenetrability of its content to engineers, the paper became known affectionately as "The Yellow Peril." This work, however, had profound implications not only for artillery, but throughout engineering. On the theoretical side, Wiener's work, interpreted by Norman Levinson, blended with the pioneering research of Kolmogorov in the Soviet Union to form the basis for communications theory, as well as strongly influencing modern ergodic theory and statistical mechanics. As explained in another section, its influence spread through physics.

Questions of how to control engineering processes abound. The origins of control theory lie deeply within the variational calculus. Its early formulations relied on methods that came directly from that part of mathematics: the Nyquist stability criterion, the Wiener filter, the Pontryagin maximum principle, the Kalman filter, and probability theory. Present research includes understanding systems governed by the heat or wave equations, such as power transmissions networks, telephone networks, chemical processing complexes, large systems of coupled electrical or mechanical devices, etc. Questions raised by robotics—including constrained motion, response to signals, etc.—all fall in this domain.

A related scientific problem is understanding the nature of digital messages. Wiener's student Claude Shannon carried out an analysis of transmission in the presence of noise. Today we view his work as the foundation of modern information theory. It provides the theoretical basis for all telephone and data communications, and the background for the work discussed in the section on coding.

The impact of time series analysis was not limited to communications. G. Wadsworth, a colleague of Wiener, happened to carpool with a geologist named Hurley. Their casual discussions around 1950 revealed that time series analysis might be useful in the seismic exploration for oil. Developed by Wadsworth, Bryan, Robinson and Hurley, this method of Wiener's has become the standard tool for modern oil exploration! At that time they implemented the new method of analyzing sound signals reflected from the earth with the aid of desk top calculators; today naturally it is carried out on large computers. In the industry, conversion to Wiener's method is referred to as the "digital revolution." It is interesting to note that 21 oil companies supported the work on applications in the geology department at MIT. However, no industrial support was given to the pure mathematical research in the same university which made the application possible—even with such a short time scale for so important a payoff. In fact the application was neither envisaged nor dreamt about at the time of the original mathematical advance—an advance oriented toward an entirely different goal.

Solid mechanics and elasticity. Solid mechanics is the science which studies the deformation and motion of solid bodies under the action of forces. It describes the behavior of steel springs and aluminum airplane wings, of rubber tires and asphalt pavement, of muscle fiber and nylon fiber.

The mathematical apparatus for describing how a body, solid or fluid, changes shape was developed by Cauchy and refined in recent years. Every part of every body must satisfy the same equations of motion. The crucial ingredient in solid mechanics is the equation which expresses how the force intensity at any point in a body is related to the change of shape near that point. We can distinguish a rubber band from a steel band of the same size by noting that a given force produces a far greater elongation in a rubber band. Other equations distinguish the responses of air, water, paint, and tar. These equations may be inferred from experiment or derived from a fundamental model.

Elasticity treats materials that are springy, such as rubber, heart muscle and steel. The linear theory of elasticity describes small deformation of elastic bodies and is the basis for the study of structures, machines, seismic waves, etc. Plasticity treats solids, like paper clips, that do not spring back to their natural state when the forces that have deformed them are removed. It furnishes an effective theory for describing the forming of metals and determining the ultimate strength of metallic structures. Results in the nonlinear theory hold promise for detecting thresholds at which materials have qualitatively different responses to their environments.

It is important to know the strength and reliability of machine parts such as valves regulating the flow of hot radioactive liquids in an atomic energy plant. The linear theory of elasticity describes well the behavior of such bodies, except near edges and corners, where cracks can form. Studies of the singularities of solutions of the equations of solid mechanics near edges and corners, of the role of plasticity and nonlinear elasticity at such singularities, and of criteria for the onset of fracture and the propagation of cracks are being actively pursued.

Dynamical systems and fluid flow. Fluid flow plays a central role in engineering, and has provided the focus of much classical mathematical study. It is generally assumed that the motion of a viscous, incompressible fluid is described by the Navier–Stokes differential equation, and in the limit of zero viscosity by the Euler equations. A typical dimensionless parameter characterizing fluid flow is the Reynolds number, which is proportional to the fluid velocity. For small Reynolds numbers (slow speeds or highly viscous flows) the equations of Navier–Stokes lead to smooth streamlines, called laminar motions. But at higher Reynolds numbers (i.e., higher speeds or lower viscosities), these laminar flows no longer persist. While they may exist as solutions of the governing equations, they are not stable. In contrast, they are replaced by time periodic or quasi-periodic perturbations of the basic flow. Bifurcations of the solutions to the equations enter here.

Only in the last twenty years have mathematicians made significant progress on the problem of this transition and the calculation of resulting flows following an instability. At even higher Reynolds numbers, the flow becomes highly irregular and is known as turbulence. Clearly any understanding of turbulence has important consequences for aircraft design, for understanding chemical reactions, combustion and flame fronts, etc. From a mathematical point of view, these equations have proved surprisingly difficult.

Even a general proof of the existence of solutions to the Navier–Stokes equations has not been found. Understanding fully developed turbulence exceeds our grasp at this time. In spite of this fact, a tremendous amount is known about some special solutions and models.

Various statistical models of turbulence were proposed by Taylor and von Karman in the 1930s. Shortly afterward, Kolmogorov introduced locally isotropic turbulence and derived the asymptotic form $E(k) \cong k^{-5/3}$ for the dependence of the energy on wave number.

Understanding turbulent solutions to the equations, or understanding the onset of turbulence as the Reynolds number increases, is still at a preliminary stage. One approach has been to obtain a priori estimates which limit the possible singular nature of a solution. Important progress in this direction has been made over the last couple of years, in bounding the Hausdorff dimension of the singular set. Some people conjecture that the Navier–Stokes equations are dominated by the viscosity and therefore have no singularities at all, though the Euler equations for zero-viscosity flow are generally expected to have singular solutions. This is an area of ongoing study, whose mathematical resolution will be of practical note.

Bifurcation theory. Bifurcation theory began with studies by the mathematician Leonhard Euler in the middle of the eighteenth century and with Poincaré's work at the end of the nineteenth century. It includes a body of techniques for studying the solutions to nonlinear equations when their character changes discontinuously as parameters in the equations cross certain thresholds. Often this occurs at particular parameter values when the equations first have nonunique (multiple) solutions. Buckling and fluttering instabilities are examples of bifurcation, as are instabilities in plasmas. This sort of question was intensively cultivated in the Soviet Union and Europe in the 1950s and 1960s. Since then, bifurcation theory has undergone a remarkable renaissance. In this development methods of point set topology, algebraic topology, and algebraic geometry have been combined with analysis.

One interesting aspect of bifurcations relates to fluid flow. In particular it is the proposal that the onset of turbulence can be described by the mathematics of successive bifurcations, leading to a transition to chaos. Several different pictures have been proposed, some involving a small number of bifurcations, others using infinitely many.

The bifurcations of iterates of quadratic maps of the unit interval $[-1, 1]$ into itself are the basis for one theoretical picture of the onset of turbulence. In certain situations, corroborating experimental evidence supports this picture. Such bifurcation problems were studied by Ulam and von Neumann in the 1940s; they even appear in earlier work of Volterra who was asked by the Norwegian government to develop a theory of populations of fish. Today, the mathematical properties of such bifurcations are also related to problems in ergodic theory, continued fraction expansions, Kleinian groups, and topology. Surprisingly, they are also related to phase cell localization and the renormalization group in mathematics and physics.

Bifurcations provide a model of chaotic behavior in a deterministic system. New universal constants have been discovered which are associated with the limit of successive bifurcations; these numbers can be measured both in numerical simulations and also in certain actual physics experiments. Numerical evidence for the existence of these universal numbers was discovered around 1976; it has recently been proved as a mathematical result—using both renormalization group ideas and a computer-assisted proof. A large mathematical literature on these problems is developing at the present time, and the interplay between the new mathematical discoveries and related phenomena in such problems as fluid flow, chemical reactions, or stellar dynamos fascinates many people.

We can hope that recent advances in the qualitative theory of differential equations with the ideas of strange attractors, the abstract mathematics of fractals and the use of super computers will lead to progress. We also look forward to new mathematical ideas to help in understanding the Navier–Stokes equations, both for fundamental reasons and also because of their technological importance.

Transonic flow and shock waves. An example of the profound interaction of mathematics and practical technology can be found in the development of methods, stimulated by the needs of aircraft designers, to calculate transonic flows. In practical terms, one models supersonic flight and shock waves. The approach depends for background on studies of partial differential equations by Tricomi in the 1920s, followed by the theoretical analysis of the numerical solutions to elliptic and hyperbolic differential equations by Courant, Friedrichs, Lewy and others. Extension of their work produced good understanding of the basic physical ideas of transonic flow, but progress was limited by the inability to calculate.

It had long been known, however, that incompressible flow theory does not explain the phenomena of high speed gas flow; rather one must use the partial differential equations of compressible gas dynamics. These equations are locally elliptic for subsonic flow and locally hyperbolic for supersonic flow; in both cases they are strikingly nonlinear.

When viscous effects are present, one must study the full nonlinear equations of Navier and Stokes, as well as other, more detailed models. Often viscous effects are confined to thin layers outside of which the fluid can be treated as inviscid. One major advance in modern applied mathematics is the theory of boundary layers, a brilliant invention in 1904 by Prandtl. He simplified the effects of viscosity without sacrificing the essential features of the flow model. The mathematical development of boundary layer theory has had profound effects in many branches of pure and applied science and enables us to come to grips with numerous phenomena in which the effects of viscosity (or other similar phenomena) are essentially restricted to well-defined regions. Indeed the study of transition effects in thin layers influences entire areas of engineering.

The advent of advanced computer calculation made it possible to include discontinuities (shocks) in the numerical algorithms. This, together with the development of new difference schemes, has made possible practical calculations. One can now obtain shock-free airfoil designs and simulate wind tunnel testing. Large computer codes based on these ideas are used by the aircraft companies on a regular basis.

Combustion theory and chemical reactions. The theory of reactive flow, or combustion theory in gases, includes all of fluid mechanics and adds an extra complication as well: the interaction of fluid flow with chemical reactions. Chemical reactions in a fluid flow change its essential characteristics. The release of heat due to exothermic reactions may cause a flow to be unstable; and the types of instability which arise may be of a different nature from ordinary fluid dynamical instabilities.

The problem of slow flames (deflagrations) in a gas, as well as theories of detonations, i.e. fluid mechanical shocks, are two areas of current research. In the theory of nuclear reactors, related issues point to models of critical size and thermal runaway. Equally important are techniques for incorporating the relevant chemistry into the mathematical analysis. More generally, chemical reactor theory accounts for diffusion and reaction, but typically includes no compressible fluid mechanical effects. Chaotic behavior in dynamical systems has found its way into chemical reaction theory. Chaotic regimes, in fact, have been experimentally and computationally found in the Belousov–Zhabotinskii reaction and other oscillatory reactions. Chaos occurs as a limit of successive bifurcations of periodic motions as the chemical concentrations or flow rates vary. The theory is strikingly similar to the mathematics of the bifurcation model for the onset of turbulence above.

Integral transforms. The Fourier transform is a special case of the general notion of an integral transform, of profound importance in physics and engineering as well as mathematics. The linear transform T relates a function $f(x)$ to its transform $(Tf)(x)$ by the formula

$$(Tf)(x) = \int K(x, y) f(y)\, dy.$$

Here $K(x, y)$ is a function which characterizes the particular transform.

Exactly when such ideas originated is not certain, but L. Euler used such a transform in 1737 to solve a differential equation. The general method was developed in the early 1800s by Gauss, Fourier, Dirichlet, Laplace and others.

The same transform studied by Euler appeared as a central tool in Laplace's classical book on probability, published in 1812. There he cast probability theory in a form more or less unchanged until the twentieth century. The transform has come to be known by his name.

The Laplace transform did not receive widespread popularity in engineering until it was rediscovered in a somewhat different guise by Heaviside toward the end of the 19th century. Faced with the practical problem of understanding the transmission and attentuation of waves in the trans-Atlantic cable (laid in 1866), Heaviside invented the "operational calculus." This powerful method solved many hitherto intractable problems in electrical engineering. Some years later, it was realized to be an aspect of Laplace transform theory, which every undergraduate engineer and scientist now studies.

A natural development of Fourier, Laplace, and Heaviside transform theory is to extend the class of functions on which they are defined. In physics, Dirac had already used such a notion with his "delta function," but no general mathematical theory existed. This led in the 1950s to the theory of distributions (developed by Schwartz, Gelfand and others), and has provided the basic tool for the modern theory of partial differential equations.

A generalization in another direction is the integral transform, generally attributed to Funk and Radon about 1916–1917. In particular, their transform of a function $f(x)$ defined on a plane is the integral of f over a line l, namely

$$(Tf)(l) = \int_l f(x)\, de.$$

Here de is the element of length on l. The original function f can be reconstructed from its Radon transform. The transform has been generalized to a transform between two homogeneous spaces of a given group, and has provided invaluable insight both in analysis and in geometry.

Roughly 60 years after the original work above, the physicist Cormack wrote a paper entitled, "Representation of a function by its line integrals." His basic problem was to understand how to reconstruct an image from an X-ray (or radioastronomy) measurement. The practical development of this idea led to computer assisted tomography, or CT scan, and was recognized in the 1979 Nobel prize for medicine. In actually building a CT scanner, one implements in a microprocessor the fastest possible convolution transform algorithm, a problem closely related to the Fourier transform algorithms described in the section on Fourier. One might expect this from the unity of mathematics.

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